Here, we will demonstrate the principle using a simple but still useful special case: The fast, discrete, type III, cosine transform. This transform is defined as

$$\mathrm{DCT}_n^{\mathrm{III}} = \left(\cos(\frac{\pi}{n}(i+\frac{1}{2})k)\right) \quad \text{for i=0 to n-1 and k=0 to n-1.} \tag{47}$$

It is used in section 11.3 for $n = 18$ and in section 11.4 for $n = 6$ to synthesize sequences of subband samples.

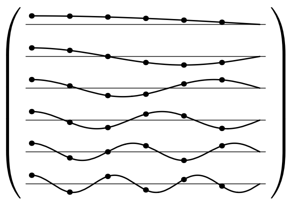
In this section, we keep the exposition as simple and short as possible and discuss only the small matrix $\mathrm{DCT}_6^{\mathrm{III}}$. For this matrix, we derive a matrix decomposition and use it, in appendix A.5, to generate efficient C code.

The matrix $\mathrm{DCT}_6^{\mathrm{III}}$ looks like this:

$$\mathrm{DCT}_6^{\mathrm{III}} = \begin{pmatrix} \cos(0) & \cos(\frac{\pi}{12}) & \cos(\frac{\pi}{6}) & \cos(\frac{\pi}{4}) & \cos(\frac{\pi}{3}) & \cos(\frac{5\pi}{12}) \\ \cos(0) & \cos(\frac{\pi}{4}) & \cos(\frac{\pi}{2}) & \cos(\frac{3\pi}{4}) & \cos(\pi) & \cos(\frac{5\pi}{4}) \\ \cos(0) & \cos(\frac{5\pi}{12}) & \cos(\frac{5\pi}{6}) & \cos(\frac{5\pi}{4}) & \cos(\frac{5\pi}{3}) & \cos(\frac{25\pi}{12}) \\ \cos(0) & \cos(\frac{7\pi}{12}) & \cos(\frac{7\pi}{6}) & \cos(\frac{7\pi}{4}) & \cos(\frac{7\pi}{3}) & \cos(\frac{35\pi}{12}) \\ \cos(0) & \cos(\frac{3\pi}{4}) & \cos(\frac{3\pi}{2}) & \cos(\frac{9\pi}{4}) & \cos(3\pi) & \cos(\frac{15\pi}{4}) \\ \cos(0) & \cos(\frac{11\pi}{12}) & \cos(\frac{11\pi}{6}) & \cos(\frac{11\pi}{4}) & \cos(\frac{11\pi}{3}) & \cos(\frac{55\pi}{12}) \end{pmatrix}$$

To apply an $n \times n$ matrix to a vector, in general n^2 multiplications and $n(n-1)$ additions are required. In this special case, because $\cos(0) = 1$, $\cos(\pi/2) = 0$ and $\cos(\pi) = -1$), we have only 26 multiplications and 28 additions. Still far too much.

The key to an efficient algorithm is the observation that the rows (and columns) of the matrix have a periodic structure. To make this structure visible, we show the same matrix again using graphs.

$\mathrm{DCT}_6^{\mathrm{III}} =$ 

Here, we discover for example that the first row $\boldsymbol{v}_1$ and the last row $\boldsymbol{v}_6$ have the same entries except for the sign, which is always positive in the first row, and alternates in the second row. The sum of the first and the last row has therefore only three non zero elements, because elements with opposite sign cancel each other. Similarly the difference of both rows as only three non zero elements, because elements with the same sign cancel each other.

$\boldsymbol{v}_1 + \boldsymbol{v}_6 =$  $\boldsymbol{v}_1 - \boldsymbol{v}_6 =$ 

Because of these zeros the scalar product with $\boldsymbol{v}_1 + \boldsymbol{v}6$ and $\boldsymbol{v}_1 - \boldsymbol{v}6$ can be computed twice as fast as the scalar product with $\boldsymbol{v}1$ and $\boldsymbol{v}6$. On the other hand, we can obtain

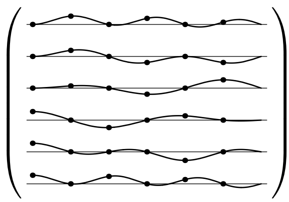
$\boldsymbol{v}_1\boldsymbol{z}$ and $\boldsymbol{v}_6\boldsymbol{z}$ easily from $\frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}6)\boldsymbol{z}$ and $\frac{1}{2}(\boldsymbol{v}_1 - \boldsymbol{v}6)\boldsymbol{z}$ using one addition and one subtraction.

Closer inspection of $\mathrm{DCT}_6^{\mathrm{III}}$ reveals that the same trick works also for rows 2 and 5, and rows 3 and 4. We package the addition and subtraction operation for each pair of rows in the following matrix:

$$\boldsymbol{Q}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and have $\mathrm{DCT}_6^{\mathrm{III}} = \boldsymbol{Q}_1\boldsymbol{Q}_1^{-1}\mathrm{DCT}_6^{\mathrm{III}}$ with

$$\boldsymbol{Q}_1^{-1}\mathrm{DCT}_6^{\mathrm{III}} = \begin{pmatrix} 0 & \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1+\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{-1+\sqrt{3}}{2\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1+\sqrt{3}}{2\sqrt{2}} \\ 1 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} =$$



We rearrange the columns by applying a permutation matrix

$$\boldsymbol{P}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

to get $\mathrm{DCT}_6^{\mathrm{III}} = \boldsymbol{Q}_1\big(\boldsymbol{Q}_1^{-1}\mathrm{DCT}_6^{\mathrm{III}}\boldsymbol{P}_1^{-1}\big)\boldsymbol{P}_1$ with

$$\boldsymbol{Q}_1^{-1}\mathrm{DCT}_6^{\mathrm{III}}\boldsymbol{P}_1^{-1} = \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{-1+\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

In summary: We have reduced the computational cost of a matrix multiplication from n^2 to $2 * (n/2)^2 + n$ by replacing the multiplication with an $n \times n$ matrix by a permutation $\boldsymbol{P}_1$ (no computational cost), two matrix multiplications of size $(n/2) \times$